Dispersion in Laminar Tube Flow at Low Peclet Numbers or Short Times

A new asymptotic solution is developed for the dispersion of a passive solute in a Newtonian fluid in fully-developed laminar flow through a straight circular tube. A perturbation solution to the convective dispersion problem is constructed by utilizing the pure diffusion solution as an initial approximation in an iterative successive approximation procedure. The derived perturbation solution is shown to be valid at both short and long times for sufficiently low values of the Peclet number and is also valid at any value of the Peclet number for sufficiently small values of time.

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Introduction

Convective dispersion plays an important role in many engineering transport problems. Consequently, it is not surprising that the dispersion of a passive solute in a Newtonian fluid in fully-developed laminar flow through a straight circular tube has been the subject of a large number of theoretical investigations. Although the boundary value problem describing this dispersion process is linear, no simple, exact, analytical solution has been developed for the local concentration distribution or for the mean concentration at a given axial location. The theoretical studies that have been carried out for this important problem can conveniently be divided into three general categories: numerical solutions, exact series solutions, and asymptotic solutions.

In the first category of solutions, finite-difference solutions have been reported by Bailey and Gogarty (1962), Ananthakrishnan et al. (1965), Gill and Ananthakrishnan (1967), and Mayock et al. (1980). In addition, an orthogonal collocation procedure has been utilized by Wang and Stewart (1983). Numerical solutions can of course be very accurate, but they are not convenient replacements for analytical solutions since a separate solution is required for each different set of conditions.

Exact series solutions have been proposed by Gill (1967), Gill and Ananthakrishnan (1967), Gill and Sankarasubramanian (1970), Tseng and Besant (1970, 1972), Yu (1976, 1979, 1981), and De Gance and Johns (1978a, b, 1980). These series solutions achieve the status of exact solutions to the problem as the number of terms in the series becomes very large. Perhaps the most comprehensive results of this type are those reported by Yu who solved the diffusion-convection problem by using a Bessel function series in conjunction with an appropriate truncation procedure for the series solution.

The computation of the terms of the series requires a significant numerical effort, and, hence, this series solution is not particularly convenient, Yu, however, has presented a large number of computed results for the laminar convective dispersion problem which can be used to establish the range of validity of asymptotic solutions. Tseng and Besant used a Bessel function series and an eigenvalue-eigenvector approach to construct a solution to the laminar dispersion problem. Yu (1981) has shown that his approach is essentially equivalent to the method used by Tseng and Besant. Consequently, the method of Tseng and Besant, like that of Yu, requires a significant computational effort in the evaluation of the series solution. Furthermore, Yu has questioned the computational accuracy of the method Tseng and Besant used to evaluate the eigenvalues and eigenvectors. Gill and his coworkers developed a series solution for the local solute concentration in the tube in terms of the mean concentration and of axial derivatives of the mean concentration. It has been shown by Gill and Subramanian (1980) and Yu (1980) that the approach of Gill and his coworkers is equivalent to the solution method that Yu used. Gill presented a truncated version of the series solution that is necessarily of restricted utility since it is characterized by what can conveniently be called the symmetry property.

A solution of the laminar dispersion problem has the symmetry property if it yields an average concentration distribution which is symmetric about the origin of the moving coordinate system at all times, if the initial distribution is symmetric. As will be noted below, the truncated solution form of Gill can be considered to be an asymptotic solution to the dispersion prob-

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lem, valid at sufficiently long times. De Gance and Johns expanded the local solute concentration in terms of a series of Hermite functions in the axial coordinate, but explicit results for the mean concentration were not presented because of the slow convergence of the proposed series. A dispersion approximation was used to provide an estimate to the slowly converging sum.

Asymptotic solutions have been constructed to the laminar dispersion problem for short and long times and for low and high values of the Peclet number. The first solution of this type appears to be the one constructed by Taylor (1953, 1954) and Aris (1956). It is generally recognized that the Taylor-Aris solution is valid asymptotically at sufficiently large values of time. Yu (1979, 1981) compared results computed from his series solution with the predictions of the Taylor-Aris solution and concluded that this solution is valid, for all practical purposes, for values of the dimensionless time, t, greater than 0.7. The Taylor-Aris solution, like the truncated solution of Gill and coworkers, has the symmetry property. Both solutions are asymptotically valid at sufficiently long times, but, as shown by Yu (1981), the Gill solution is applicable at smaller times than are required for the Taylor-Aris solution. Chatwin (1970) also obtained a long-time solution by formulating an asymptotic series for the solute concentration which illustrates how an asymmetric solution for the mean concentration approaches the symmetric form as $t \rightarrow \infty$. Chatwin claimed that his asymptotic series was valid for $t > 0.2$, but a comparison of the solution of Yu (1979, 1981) with that of Chatwin indicates that somewhat larger values of dimensionless time are needed for the solution of Chatwin to yield satisfactory mean concentration profiles.

Short-time asymptotic solutions to the laminar dispersion problem have been proposed by Lighthill (1966), Chatwin (1976), and Hunt (1977). The solution of Lighthill is restricted to high Peclet numbers as well as to small values of the dimensionless time. The analysis of Lighthill applies to situations for which the injected solute does not interact with the tube wall, and Lighthill thus estimated that his short-time solution should be valid for dimensionless times t less than about 0.1. However, from comparisons carried out by Yu (1979, 1981), it appears that the Lighthill solution does not perform particularly well even for $t = 0.02$. Chatwin (1976) extended the Lighthill solution to all values of the Peclet number by keeping the axial diffusion term in the continuity equation for the solute concentration. Again the analysis is valid only if the solute does not interact with the tube wall.

Furthermore, determination of the mean concentration of the solute at particular values of the axial position and time requires the numerical evaluation of two improper integrals. Hunt (1977) used a singular perturbation method to construct a solution that appears to be asymptotically valid at small times for flow with large Peclet numbers. However, Yu (1979, 1981) has shown that the solution derived by Hunt is not generally quantitatively correct at small values of time. Finally, Fife and Nicholes (1975) presented a method for deriving solutions for low values of the Peclet number, but they derive no new results for circular tube flow using this method.

The objective of this study is to formulate a new asymptotic solution which will complement the solutions that are valid asymptotically at sufficiently large values of time (the Taylor-Aris and Gill solutions). A perturbation solution to the dispersion problem is constructed by utilizing a successive approximation method with the pure diffusion solution as the basic initial approximation in the iteration procedure. It is shown that the derived perturbation solution is valid at both short and long times for sufficiently low values of the Peclet number and is also valid at any value of the Peclet number for sufficiently small values of time. The equations and solution method are formulated in the second section of the paper, and the perturbation solution is presented in the third section of the paper. The properties and range of validity of the solution are investigated in the final section of the paper.

Problem Formulation

We consider the laminar, axisymmetric, isothermal flow of an incompressible, two-component Newtonian fluid in an infinitely long circular tube. A finite amount of additional solute is injected into the system, but the amount is sufficiently small that all physical properties in the system are effectively constant. Furthermore, the solute is passive in the sense that it undergoes no chemical reactions. The species continuity equation for the solute can be written in dimensionless form as follows:

$$\frac{\partial C}{\partial t} + Pe\left(\frac{1}{2} - r^2\right)\frac{\partial C}{\partial \lambda} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{\partial^2 C}{\partial \lambda^2} \quad (1)$$

where the axial coordinate λ is defined such that it has an origin moving with the average velocity of the flow. The boundary conditions for the problem can be expressed as follows:

$$\frac{\partial C}{\partial r}(0, \lambda, t) = 0 \quad (2)$$

$$\frac{\partial C}{\partial r}(1, \lambda, t) = 0 \quad (3)$$

$$C(r, -\infty, t) = 0 \quad (4)$$

$$C(r, +\infty, t) = 0 \quad (5)$$

In the present paper, the discussion is limited to the development of a solution for a concentrated initial solute input distributed uniformly over the cross section of the tube at the origin $z^* = 0$ of the stationary coordinate system. For this concentrated injection of solute, the initial condition can be written as follows:

$$C(r, \lambda, 0) = \delta(\lambda) \quad (6)$$

An asymptotic solution to the above set of equations is formulated using a perturbation method. An iteration procedure is used to find successive approximations upon a basic solution; in this case, the pure diffusion solution is taken to be the initial approximation in the iterative procedure. It can be expected that the perturbation expansion, which is generated by this procedure, should be valid whenever the ratio of convective terms to diffusive terms in the species continuity equation is suitably small. Ananthakrishnan et al. (1965) showed that diffusion should dominate convection when the quantity $Pe\sqrt{t}$ is sufficiently small. Consequently, it is reasonable to expect that the perturbation expansion derived from the proposed iteration procedure should yield reasonable results for both short and long times for sufficiently low values of the Peclet number and also for any value of the Peclet number for sufficiently small values of time.

The initial or zero-order approximation to C (which is denoted as C_0) is obtained by neglecting the convective term altogether in Eq. 1. Consequently, C_0 is obtained by solving the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_0}{\partial r}\right) + \frac{\partial^2 C_0}{\partial \lambda^2} - \frac{\partial C_0}{\partial t} = 0 \tag{7}$$

subject to Eqs. 2–6. To calculate the first-order effects of convection, the zero-order solution is used to evaluate the convective term in Eq. 1, and the resulting equation is then solved for the first-order approximation C_1, subject again to Eqs. 2–6. Generally, then, we solve for the complete solution at each stage, using the following equation for the nth order approximation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_n}{\partial r}\right) + \frac{\partial^2 C_n}{\partial \lambda^2} - \frac{\partial C_n}{\partial t} = Pe\left(\frac{1}{2} - r^2\right)\frac{\partial C_{n-1}}{\partial \lambda} \tag{8}$$

It is clear that the full boundary conditions, Eqs. 2–6, must be imposed at each stage of the iteration process.

The solution of Eq. 8 subject to Eqs. 2–6 can conveniently be constructed using the method of Green's functions (Stagkold, 1968). The solution to Eq. 8 can thus be written as follows for $n \geqq 1$

$$C_n(r, \lambda, t) = \int_0^t \int_0^1 \int_{-\infty}^{\infty} q_{n-1}(r_0, \lambda_0, t_0) g(r, \lambda, t | r_0, \lambda_0, t_0) r_0 \, d\lambda_0 \, dr_0 \, dt_0 + \int_0^1 g(r, \lambda, t | r_0, 0, 0) r_0 \, dr_0 \tag{9}$$

where

$$q_{n-1}(r, \lambda, t) = -Pe\left(\frac{1}{2} - r^2\right)\frac{\partial C_{n-1}}{\partial \lambda} \tag{10}$$

and where $g(r, \lambda, t | r_0, \lambda_0, t_0)$ is given by the expression

$$g(r, \lambda, t | r_0, \lambda_0, t_0) = \frac{H(t - t_0)\exp\left[-\frac{(\lambda - \lambda_0)^2}{4(t - t_0)}\right]}{\sqrt{\pi}\sqrt{t - t_0}} \cdot \sum_{n=0}^{\infty} \frac{J_0(\alpha_n r_0) J_0(\alpha_n r) e^{-\alpha_n^2 (t - t_0)}}{J_0^2(\alpha_n)} \tag{11}$$

where the α_n are the roots (including 0) of

$$J_1(\alpha_n) = 0 \tag{12}$$

Equations 9 and 11 thus provide a convenient vehicle for the development of the perturbation series. This is illustrated in the next section. Finally, it is necessary to define a radial area average concentration at each axial location since this is usually the quantity of prime importance in dispersion theories. In this paper, the average solute concentration is defined as follows:

$$\overline{C} = \int_0^1 Cr \, dr \tag{13}$$

Construction of Perturbation Solution

As noted in the previous section, expressions for C_n can be obtained for $n \geqq 1$ by utilizing Eqs. 9 and 11. The solution for C_0 can also be obtained using a Green's function approach. The solutions for C_0 and for the zero-order approximation to the average concentration are simply:

$$C_0 = \frac{e^{-\lambda^2/4t}}{2\sqrt{\pi t}} = 2\overline{C}_0 \tag{14}$$

Equation 14 is the pure diffusion solution. The solution to Eq. 8 for $n = 1$ can be obtained from Eqs. 9 and 11 by utilizing the following expression for q_0:

$$q_0 = \frac{Pe\left(\frac{1}{2} - r^2\right)\lambda e^{-\lambda^2/4t}}{4\sqrt{\pi} t^{3/2}} \tag{15}$$

The solution for C_1 can thus be expressed as

$$C_1 = C_0 - \frac{Pe\lambda e^{-\lambda^2/4t}}{\sqrt{\pi} t^{3/2}} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)[1 - e^{-\alpha_n^2 t}]}{J_0(\alpha_n)\alpha_n^4} \tag{16}$$

and, furthermore, it follows immediately that

$$\overline{C}_1 = \overline{C}_0 \tag{17}$$

The solution to Eq. 8 for $n = 2$ can be obtained from Eqs. 9 and 11 by introducing the following equations for q_1:

$$q_1 = q_0 + \frac{Pe^2 e^{-\lambda^2/4t}}{\sqrt{\pi}}\left(\frac{1}{2} - r^2\right)\left[\frac{2}{t^{3/2}} - \frac{\lambda^2}{t^{5/2}}\right] f(r, t) \tag{18}$$

$$f(r, t) = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{2J_0(\alpha_n)\,\alpha_n^4}[1 - e^{-\alpha_n^2 t}] \tag{19}$$

The solution for C_2 can thus be expressed as follows

$$C_2 = C_1 + \frac{Pe^2 e^{-\lambda^2/4t}}{\sqrt{\pi}}\left[\frac{2}{t^{3/2}} - \frac{\lambda^2}{t^{5/2}}\right] F(r, t) \tag{20}$$

where

$$F(r, t) = -\sum_{n=1}^{\infty} \frac{2}{\alpha_n^6}\left[t + \frac{e^{-\alpha_n^2 t} - 1}{\alpha_n^2}\right] + \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{12\alpha_n^4 J_0(\alpha_n)}\left[\frac{1 - e^{-\alpha_n^2 t}}{\alpha_n^2} - te^{-\alpha_n^2 t}\right] - \sum_{n=1}^{\infty}\sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{J_0(\alpha_m r)\, C_{mn}}{\alpha_n^4 J_0(\alpha_n) J_0^2(\alpha_m)} \cdot \left[\frac{1 - e^{-\alpha_m^2 t}}{\alpha_m^2} - \frac{e^{-\alpha_n^2 t} - e^{-\alpha_m^2 t}}{\alpha_m^2 - \alpha_n^2}\right] \tag{21}$$

$$C_{mn} = \int_0^1 r^3 J_0(\alpha_m r) J_0(\alpha_n r) dr \tag{22}$$

Consequently, the following expression can be derived for $\overline{C}_2$:

$$\overline{C}_2 = \overline{C}_0\left[1 - Pe^2\left(\frac{8}{t} - \frac{4\lambda^2}{t^2}\right)\sum_{n=1}^{\infty}\frac{1}{\alpha_n^6}\left(t + \frac{e^{-\alpha_n^2 t} - 1}{\alpha_n^2}\right)\right] \quad (23)$$

In a similar manner, the following solution can be derived for $\overline{C}_3$

$$\overline{C}_3 = \frac{e^{-(\lambda^2/4t)}}{4\sqrt{\pi t}}\left[1 - Pe^2\left(8 - \frac{4\lambda^2}{t}\right)A + Pe^3\left(\frac{2\lambda^3}{t^2} - \frac{12\lambda}{t}\right)(B - E)\right] \quad (24)$$

where

$$A(t) = \sum_{n=1}^{\infty}\frac{\left[1 + \dfrac{e^{-\alpha_n^2 t} - 1}{\alpha_n^2 t}\right]}{\alpha_n^6} \quad (25)$$

$$B(t) = \sum_{n=1}^{\infty}\frac{\left[\dfrac{1}{\alpha_n^2} + \dfrac{2(e^{-\alpha_n^2 t} - 1)}{\alpha_n^4 t} + \dfrac{e^{-\alpha_n^2 t}}{\alpha_n^2}\right]}{6\alpha_n^6} \quad (26)$$

$$E(t) = \sum_{n=1}^{\infty}\sum_{\substack{m=1 \\ m\neq n}}^{\infty}\frac{2C_{mn}}{\alpha_m^2 J_0(\alpha_n)\, J_0(\alpha_m)\, \alpha_n^4} \times\left[\frac{1}{\alpha_m^2} + \frac{e^{-\alpha_m^2 t} - 1}{\alpha_m^4 t} + \frac{\alpha_m^2 e^{-\alpha_n^2 t} - \alpha_n^2 e^{-\alpha_m^2 t}}{\alpha_m^2\alpha_n^2(\alpha_m^2 - \alpha_n^2)t} - \frac{1}{t\alpha_n^2\alpha_m^2}\right] \quad (27)$$

The evaluation of $\overline{C}_3$ as a function of λ and t is facilitated by utilization of the following result:

$$C_{mn} = \frac{2(\alpha_m^2 + \alpha_n^2)\, J_0(\alpha_m)\, J_0(\alpha_n)}{(\alpha_m^2 - \alpha_n^2)^2}\quad m \neq n \quad (28)$$

Clearly, the calculation of $\overline{C}_3$ from Eq. 24 involves only simple series summation.

Higher order approximations (with $n \geqq 4$) to the average concentration $\overline{C}$ can of course be derived, but the labor involved rapidly becomes excessive. It is, however, possible to improve the utility of the perturbation series given by Eq. 24 by increasing the accuracy of the series and by extending its range of validity. This can be done in this case by introducing an Euler-type transformation (Van Dyke, 1974) for the Peclet number. Introduction of a modified Peclet number

$$\overline{Pe}^2(t) = \frac{8Pe^2A}{1 + 8Pe^2A} \quad (29)$$

leads to the following modified version of the perturbation series for $\overline{C}_3$:

$$\overline{C}_3 = \frac{e^{-(\lambda^2/4t)}}{4\sqrt{\pi t}}\left[1 - \left(1 - \frac{\lambda^2}{2t}\right)\overline{Pe}^2 + \frac{(B - E)}{A^{3/2}8^{3/2}}\left(\frac{2\lambda^3}{t^2} - \frac{12\lambda}{t}\right)\overline{Pe}^3\right] \quad (30)$$

It can be shown that the asymptotic series given by Eq. 30 has superior convergence properties than the perturbation series given by Eq. 24. An evaluation of the properties and range of validity of Eq. 24 and Eq. 30 is carried out in the next section using results computed by Yu (1981) from his exact series solution. The present solution is also compared to the Taylor-Aris and Gill long time asymptotic solutions and to the pure convection solution. The pure convection solution can be derived by solving the equation

$$\frac{\partial C}{\partial t} + Pe\left(\frac{1}{2} - r^2\right)\frac{\partial C}{\partial \lambda} = 0 \quad (31)$$

subject to the initial condition given by Eq. 6. The solution is simply

$$C = \delta[\lambda - Pe(1/2 - r^2)\, t] \quad (32)$$

and, hence, the average concentration $\overline{C}$ is described by the following set of equations:

$$2Pe\, t\, \overline{C} = 1 \quad 1 > \frac{2\lambda}{Pet} > -1 \quad (33)$$

$$2Pe\, t\, \overline{C} = 0 \quad \left|\frac{2\lambda}{\mathrm{Pet}}\right| > 1 \quad (34)$$

In the discussion that follows, Eq. 24 will be denoted as the *basic* perturbation series, and Eq. 30 will be termed the *modified* perturbation series. Equation 24 will be used to deduce some basic properties of the perturbation series, but Eq. 30 will be utilized in all numerical calculations because of its superior convergence properties.

Evaluation of Perturbation Solution

Since a new asymptotic solution, Eq. 24 or 30, has been developed in this paper, it is necessary to evaluate its properties and its range of validity. First of all, it should be noted that this solution is of theoretical interest since it leads to an understanding of the mechanics of the dispersion process for small times, whereas much of the previous work on dispersion has given results valid asymptotically as $t \rightarrow \infty$. Thus, one result of this paper is to provide a simple small-time solution which complements the available simple large time solutions (the Taylor-Aris and Gill solutions). As noted by Chatwin (1976), small-time solutions are of interest for many important flows, and thus a solution is needed to describe how the details of the flow and of the initial solute distribution affect the dispersion process in the limit $t \rightarrow 0$.

It is possible to establish a connection between the present solution and the long-time solutions of Taylor-Aris and Gill. It is easy to show that the basic perturbation solution, Eq. 24, to order Pe^2, is identical to the Gill solution for a concentrated initial solute input (Yu, 1979), to order Pe^2, for all values of time. Consequently, for sufficiently small values of Pe, where the Pe^3 term and higher-order terms are negligible, the Gill solution is not restricted to the long-time region since the average concentration distribution effectively possesses the symmetry property for all values of time. At higher values of Pe, where the Pe^3 term begins to become important, the Gill solution does not agree with Eq. 24 since the Gill solution retains the symmetry prop-

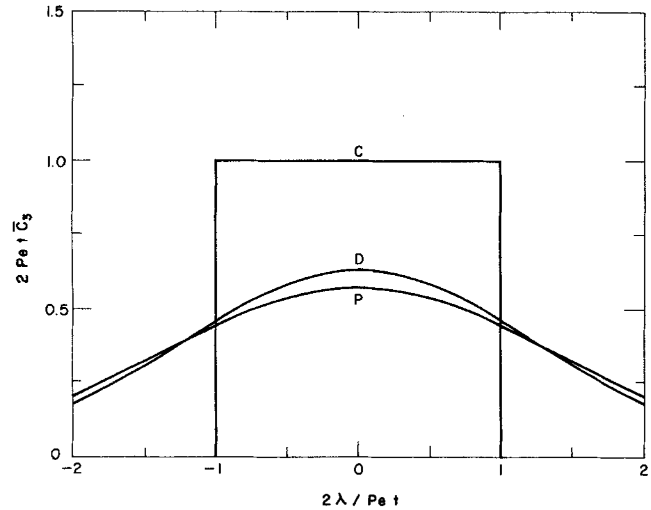

Figure 1a. Early time behavior of perturbation solution for Pe = 1,000 and t = 5 × 10^{-6}.
Curve P, perturbation solution; Curve D, diffusion solution; Curve C, convection solution.

erty whereas Eq. 24 clearly does not. Hence, in such cases, the Gill solution is restricted to long times since it cannot predict asymmetric concentration profiles that exist at shorter times. It is also possible to show that the basic perturbation solution, Eq. 24, to order Pe^2, is identical to the Taylor-Aris solution for a concentrated initial input (Yu, 1979), to order Pe^2, but only at sufficiently large values of time. Clearly, from these results, it is evident that the Gill solution is valid at smaller times than the Taylor-Aris solution, as is of course well known (Yu, 1981). Finally, the Taylor-Aris solution will be valid for all times only when the Peclet number is very small so that the term of order Pe^2 can be neglected.

One useful way of studying the mechanics of the dispersion process for small t is to examine the time dependence of the average concentration distribution for high Peclet numbers. Ananthakrishnan et al. (1965) noted that axial molecular diffusion predominates over axial convection even for large values of the Peclet number if t is sufficiently small. This type of behavior is indeed predicted by the perturbation solution developed in this paper. The behavior predicted by the modified perturbation solution, Eq. 30, at early times for Pe = 1,000 is illustrated in Figures 1a–1d. At the lowest value of time, it is evident that the predicted average concentration distribution is close to the pure diffusion solution. As time increases however, the average concentration profile computed from the perturbation solution begins to approach the solution for pure convection. For even larger values of time, an asymmetric concentration distribution exists, and ultimately the Taylor-Aris solution is achieved at sufficiently long times.

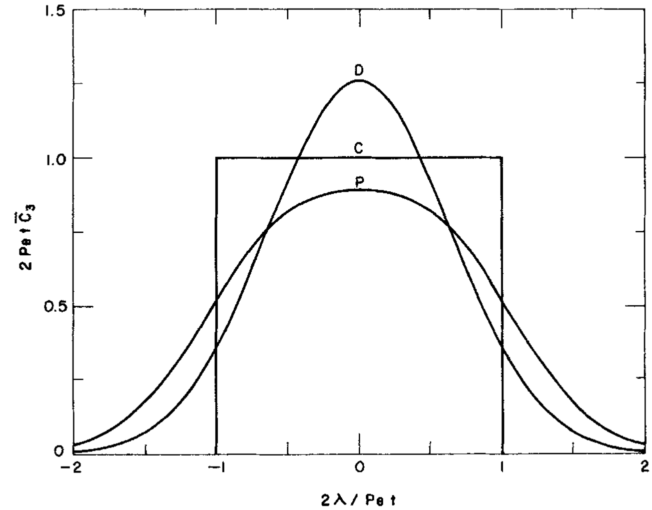

Figure 1c. t = 2 × 10^{-5}

The development of an asymmetric average concentration distribution from the initial symmetric form as time increases is illustrated in Figure 2 for Pe = 15. In this figure, the position of the maximum in the average concentration profile is plotted vs. dimensionless time. The calculations are carried out using Eq. 30. Initially, the maximum in the mean concentration distribution is at the origin of the moving coordinate system ($\lambda = 0$) before shifting to negative values of λ. Ultimately, the maxi-

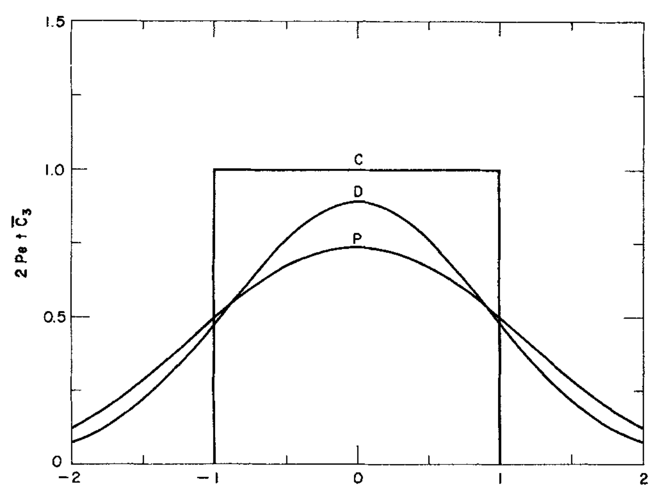

Figure 1b. t = 10^{-5}

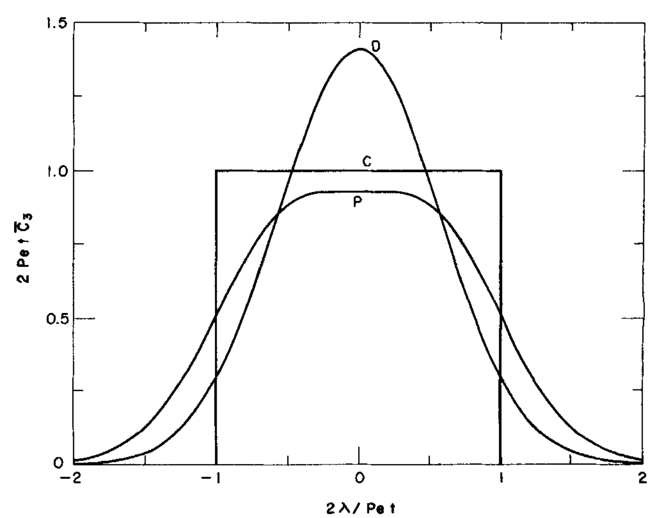

Figure 1d. t = 2.5 × 10^{-5}

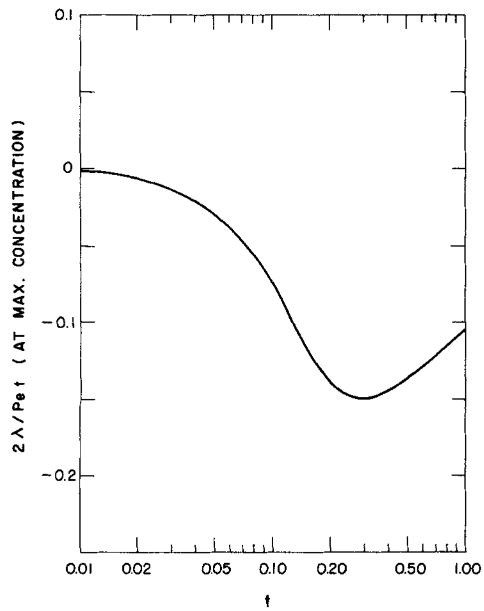

Figure 2. Time dependence of position of the maximum in the average concentration profile for *Pe* = 15.

mum returns to $\lambda = 0$. The Gill and Taylor-Aris solutions cannot of course predict such asymmetries in the concentration distribution since the maximum is at $\lambda = 0$ for all times for these asymptotic solutions. For example, the differences in average concentration distributions between the Gill solution and the modified perturbation solution, Eq. 30, are illustrated in Figure 3.

The change of the form of the average concentration distribution with increasing time can also be monitored by calculating the ratio of the amount of added solute in the region with negative values of λ relative to the mean flow position to that with positive values of λ:

$$\bar{R} = \frac{\int_{-\infty}^{0} \bar{C}\, d\lambda}{\int_{0}^{\infty} \bar{C}\, d\lambda} \tag{35}$$

Initially, for the concentrated input, $\bar{R} = 1$, but $\bar{R}$ should increase with increasing time, reach a maximum value, and then decrease toward unity as the average concentration profile eventually becomes symmetric again about the plane moving with the mean flow velocity. This type of behavior is illustrated in Figure 4 for $Pe = 15$. This figure is based on the following equations which are derived by substituting Eq. 30 into Eq. 35:

$$\bar{R} = \frac{\frac{1}{4} + \psi}{\frac{1}{4} - \psi} \tag{36}$$

$$\psi = \frac{2Pe^3(B - E)}{\sqrt{\pi t}\,(1 + 8Pe^2 A)^{3/2}} \tag{37}$$

The practical utility of the new perturbation solution of course depends on how wide are the ranges of time and Peclet number for which the solution provides an adequate estimate of the average concentration distribution in the tube. Chatwin (1970) suggested that one way of assessing the validity of asymptotic solutions is to compare the first few axial integral moments computed from the asymptotic solution with the *exact* values which can be determined by solving the appropriate moment equations. A convenient set of time-dependent moments can be defined in the following manner:

$$I_p = \int_0^1 \int_{-\infty}^{\infty} \lambda^p C r\, d\lambda\, dr \tag{38}$$

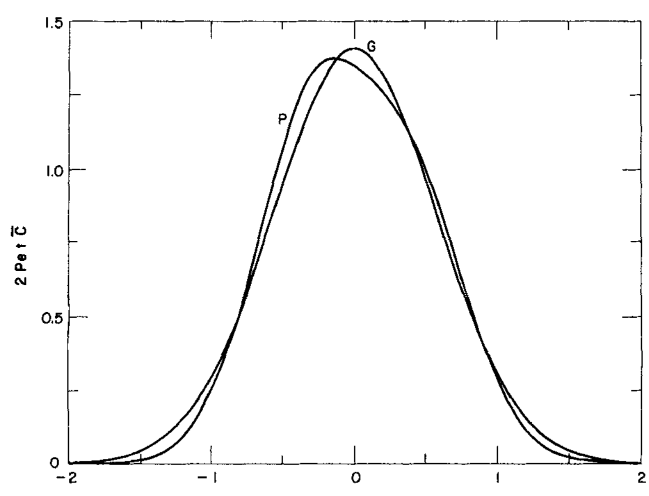

Figure 3. Comparison of average concentration distributions for perturbation solution and for Gill solution for *Pe* = 15 and *t* = 0.2.

Curve P, perturbation solution; Curve G, Gill solution.

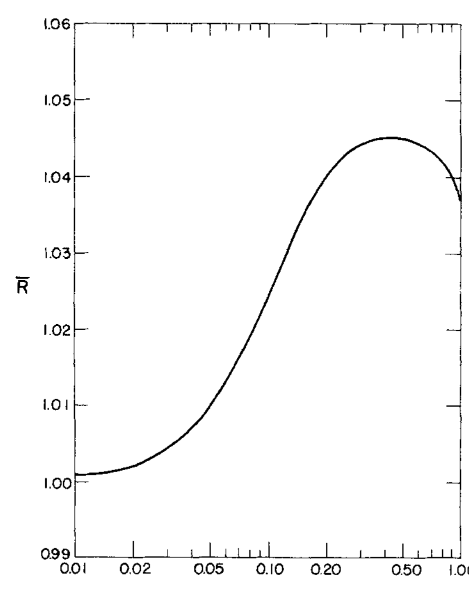

Figure 4. Time dependence of $\bar{R}$ for *Pe* = 15.

and we are interested here in $p = 0, 1, 2$, and 3. From the basic perturbation solution, Eq. 24, it is easy to derive the following results by simple integration:

$$I_0 = \tfrac{1}{2} \tag{39}$$

$$I_1 = 0 \tag{40}$$

$$I_2 = t + \frac{Pe^2 t}{192} - \frac{Pe^2}{2{,}880} + 16Pe^2 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 t}}{\alpha_n^8} \tag{41}$$

$$I_3 = 48Pe^3(B - E)t \tag{42}$$

The exact solutions to the appropriate moment equations produce precisely the same results for I_0, I_1, I_2, and I_3. The values of I_0, I_1, and I_2 for the exact solutions of the moment equations are well known (Aris, 1956; Chatwin, 1970; Wang and Stewart, 1983), and a derivation of the correct value for I_3 will be presented in a future communication. Consequently, the basic perturbation solution has the property that it produces values of the first four integral moments that agree over the entire time domain with the exact values derived from solution of the appropriate moment equations. None of the available long-time solutions has this property. The solutions with the symmetry property (the Gill and Taylor-Aris solutions) necessarily have $I_3 = 0$ for all times, and the solution of Chatwin (1970) deviates from the exact value of I_2 at short times. However, even though the basic perturbation solution has exact values for the first four moments for all values of time, it provides good predictions for the average concentration profile only for small values of time. Hence, a moment comparison can be misleading, and we thus evaluate the perturbation solution in this paper using the computed results presented by Yu (1981). All of the predictions are obtained using the modified perturbation series, Eq. 30, rather than the basic perturbation series, Eq. 24, since the former has better convergence properties.

The dimensionless peak mean concentration for various values of the Peclet number was evaluated as a function of dimensionless time from Eq. 30 and compared with the results of Yu (1981) in Figure 5. The results of Yu, which are presented as solid lines in Figure 5, were taken directly from a figure presented in his paper. At each value of Pe, there is a time interval, $0 \leqq t \leqq t_m$, over which the modified perturbation series, Eq. 30, produces a good estimate of the maximum average concentration. Here, t_m is the largest value of dimensionless time at which the perturbation series should be used to calculate the average concentration profile at a particular value of the Peclet number. For $Pe \leqq 15$, t_m is at least unity. For $Pe > 15$, the following equation provides a conservative estimate for t_m:

$$Pe\sqrt{t_m} = 5 \tag{43}$$

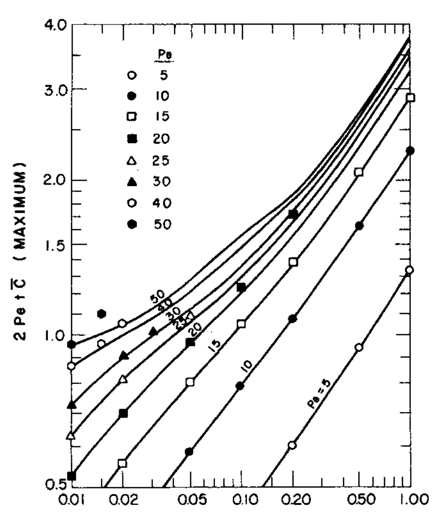

Figure 5. Dimensionless peak mean concentrations: perturbation solution vs. solution of Yu (1981).
The lines are the solution of Yu and the individual points were computed from the perturbation solution.

Clearly, the new perturbation equation is very useful at low Peclet numbers ($Pe \leqq 15$) since it is valid over a time interval which overlaps the time interval for the Taylor-Aris solution ($t \geqq 0.7$). Furthermore, it predicts the shapes of asymmetric average concentration distributions whereas the Taylor-Aris and Gill solutions predict only symmetric concentration profiles. As the Peclet number increases, the acceptable time interval for the perturbation solution decreases significantly so that there no longer is overlap with the time domain of the Taylor-Aris solution. However, the perturbation series still provides acceptable results at high values of the Peclet number for sufficiently small values of the dimensionless time. Consequently, the perturbation series solution is valid not only at all times of interest for sufficiently small values of Pe, but it provides predictions for any value of the Peclet number for sufficiently small values of time. This time interval will be of practical interest for some, but of course not all, transport processes.

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Notation

A = function defined by Eq. 25
B = function defined by Eq. 26
C = dimensionless mass density of solute = $C^* - C_0/C_1 - C_0$
C^* = mass density of solute
C_0 = mass density of solute in mixture before injection
$C_1 = C_0 + M_0/\pi R^3$
C_n = nth order approximation for C
C_{mn} = constants defined by Eq. 22
$\overline{C}$ = radial average solute concentration defined by Eq. 13
$\overline{C}_n$ = nth order approximation for $\overline{C}$
D = binary mutual diffusion coefficient
E = function defined by Eq. 27
f = function defined by Eq. 19
F = function defined by Eq. 21
g = Green's function
H = step function
I_p = moment defined by Eq. 38
J_0 = Bessel function of first kind of order zero
J_1 = Bessel function of first kind of order one
M_0 = mass of injected solute
Pe = Peclet number = RU_c/D
$\overline{Pe}$ = modified Peclet number defined by Eq. 29
q_{n-1} = function defined by Eq. 10
r = dimensionless radial distance variable = r^*/R
r^* = radial distance variable
R = radius of tube
$\overline{R}$ = ratio defined by Eq. 35
t = dimensionless time = Dt^*/R^2

t^* = time

t_m = upper limit of dimensionless time for utilization of perturbation series

U_c = velocity at center of tube

z^* = axial distance variable

Greek letters

α_n = roots of Eq. 12

δ = Dirac delta function

λ = dimensionless axial distance variable = $(z^* - U_c t^*/2)/R$

ψ = function defined by Eq. 37

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